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One-loop divergences in the two-dimensional non-anticommutative supersymmetric σ-model

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Abstract

We discuss the structure of the non-anticommutative $\mathcal{N} = 2$ non-linear σ-model in two dimensions, constructing differential operators which implement the deformed supersymmetry generators and using them to reproduce the classical action. We then compute the one-loop quantum corrections and express them in a more compact form using the differential operators.

1 Introduction

The subject of deformed quantum field theories has attracted renewed attention in recent years due to their natural appearance in string theory. Initial studies were devoted to theories on non-commutative spacetime in which the commutator of the spacetime co-ordinates becomes non-zero. More recently [1]- [9], non-anticommutative supersymmetric theories have been constructed by deforming the anticommutators of the grassman co-ordinates θ^{α} (while leaving the anticommutators of the $\overline{\theta}^{\dot{\alpha}}$ unaltered). Consequently, the anticommutators of the supersymmetry generators $\overline{Q}_{\dot{\alpha}}$ are deformed while those of the Q_{α} are unchanged. Non-anticommutative versions of the Wess-Zumino model and supersymmetric gauge theories have been formulated in four dimensions [10,11] and their renormalisability discussed [12]- [16], with explicit computations up to two loops [17] for the Wess-Zumino model and one loop for gauge theories [18]- [22].

More recently still, non-anticommutative theories in two dimensions have been considered. On the one hand non-anticommutative versions of particular non-linear σ-models have been constructed (by dimensional reduction from four dimensions) [23] and the one-loop corrections computed [24]; on the other hand a non-anticommutative version of the general $\mathcal{N}=2$ Kähler σ-model has been constructed directly in two dimensions, initially in Refs. [25,26] but then given an elegant reformulation in Refs. [27,28]. We shall predominantly follow the notation of Ref. [27], where the deformation was interpreted as a "smearing" of the Kähler potential. The undeformed $\mathcal{N}=2$ Kähler σ-model and its renormalisation were studied exhaustively in the context of string theory. It was thought for a while that its only divergences were at the one-loop level where they can be interpreted as a correction to the Kähler metric of the form of the Ricci tensor; until explicit calculations [29,30] revealed a divergence at the four-loop level.

The motivation for the present work was to investigate whether the one-loop corrections in the deformed theory as presented in Ref. [27] would exhibit a similar "smearing" as in the classical theory. It turns out that the number of one-loop diagrams in the deformed theory is enormous, at least in the component formulation in which we work; however, they can be expressed in terms of differential operators implementing the undeformed supersymmetry generators $Q_{\pm}$ (using light-cone co-ordinates in two dimensions), acting on a simpler "kernel". Now in fact, the undeformed classical action (in its component form) can be expressed simply as the product of the operators representing *all* the supersymmetry generators, $Q_{\pm}$ and $\overline{Q}_{\pm}$, acting on the Kähler potential. This inspired the hope that in the non-anticommutative case, if we could construct the operators implementing the deformed supersymmetry generators $\overline{Q}_{\pm}$, we might be able to obtain a similarly succinct form for the deformed one-loop corrections. Accordingly, we start by giving an exact construction for these operators to all orders in the deformation parameter. We then give our results for the one-loop calculation, expressed in a relatively compact form in terms of the undeformed operators for $Q_{\pm}$ acting on a kernel $\mathcal{K}$. It is then easy to see that unfortunately it is impossible to further write $\mathcal{K}$ in a shorter form using the operators representing $\overline{Q}_{\pm}$.

2 $\mathcal{N} = 2$ supersymmetry in two dimensions

In this section we set the scene for the analysis by describing in some detail the case of undeformed supersymmetry in two dimensions, focussing on the use of differential operators to implement the supersymmetry and simplify the description. In two dimensions it is convenient to use "lightcone" co-ordinates $x^{\pm}$, $\theta^{\pm}$, $\overline{\theta}^{\pm}$ (a slight abuse of terminology since in the non-anticommutative case we are obliged to work on a spacetime of Euclidean signature). We now consider a theory with a multiplet of chiral superfields $\Phi^i(x^{\pm}, \theta^{\pm}, \overline{\theta}^{\pm})$ (with components φ^i, ψ^i, F^i). We denote the conjugate fields by $\overline{\Phi}^{\bar{i}}$, $\overline{\varphi}^{\bar{i}}$, etc; though often we suppress the superscripts. The simplest model is the two-dimensional $\mathcal{N} = 2$ non-linear σ-model whose action is, in (undeformed) superspace, given by

$$S_0 = \int d^2x d^2\theta d^2\overline{\theta} K(\Phi, \overline{\Phi}) \tag{1}$$

where K is the Kähler potential.

The charges are then

$$Q_{\pm} = \frac{\partial}{\partial\theta^{\pm}}, \quad \overline{Q}_{\pm} = -\frac{\partial}{\partial\overline{\theta}^{\pm}} - i\theta^{\pm}\frac{\partial}{\partial y^{\pm}}, \tag{2}$$

where

$$y^{\pm} = x^{\pm} - i\theta^{\pm}\overline{\theta}^{\pm}. \tag{3}$$

They satisfy the algebra

$$\begin{aligned} Q_+^2 = Q_-^2 \quad &= \quad \{Q_+, Q_-\} = 0, \\ \overline{Q}_+^2 = \overline{Q}_-^2 = 0, \qquad & \qquad \{\overline{Q}_+, \overline{Q}_-\} = 0, \\ \{\overline{Q}_+, Q_+\} = -i\partial_+, \qquad & \qquad \{\overline{Q}_-, Q_-\} = -i\partial_-. \end{aligned} \tag{4}$$

The superfields have expansions in terms of component fields given by

$$\begin{aligned} \Phi &= \varphi + \theta^+\psi_+ + \theta^-\psi_- + \theta^+\theta^- F, \\ \overline{\Phi} &= \overline{\varphi} + \overline{\theta}^+\left[\overline{\psi}_+ - i\theta^+\partial_+\overline{\varphi}\right] + \overline{\theta}^-\left[\overline{\psi}_- - i\theta^-\partial_-\overline{\varphi}\right] \\ &+ \overline{\theta}^+\overline{\theta}^-\left[\overline{F} + i\theta^+\partial_+\overline{\psi}_- - i\theta^-\partial_-\overline{\psi}_+ + \theta^+\theta^-\partial_+\partial_-\overline{\varphi}\right], \end{aligned} \tag{5}$$

where the component fields are functions of $y^{\pm}$, as defined in Eq. (3). It is useful to represent the charges $Q_{\pm}$, $\overline{Q}_{\pm}$ by differential operators $q_{\pm}$, $\overline{q}^0_{\pm}$ acting on the fields, i.e.

$$[Q_{\pm}, \Phi] = q_{\pm}\Phi, \tag{6a}$$

$$[\overline{Q}_{\pm}, \Phi] = \overline{q}^0_{\pm}\Phi \tag{6b}$$

where

$$\begin{aligned} q_{\pm} &= \psi_{\pm}\frac{\partial}{\partial\varphi} \mp F\frac{\partial}{\partial\psi_{\mp}} - i\partial_{\pm}\overline{\varphi}\frac{\partial}{\partial\overline{\psi}_{\pm}} \pm i\partial_{\pm}\overline{\psi}_{\mp}\frac{\partial}{\partial\overline{F}}, \\ \overline{q}^0_{\pm} &= -\overline{\psi}_{\pm}\frac{\partial}{\partial\overline{\varphi}} \pm \overline{F}\frac{\partial}{\partial\overline{\psi}_{\mp}} + i\partial_{\pm}\varphi\frac{\partial}{\partial\psi_{\pm}} \mp i\partial_{\pm}\psi_{\mp}\frac{\partial}{\partial F}. \end{aligned} \tag{7}$$

We use the superscript "0" to denote the undeformed case; since $q_\pm$ will be unchanged in the deformed case, no superscript is needed for the unbarred operators. These operators have anticommutation properties analogous to Eq. (4), except that

$$\{\overline{q}_+^0, q_+\} = i\partial_+, \quad \{\overline{q}_-^0, q_-\} = i\partial_-. \tag{8}$$

Note the change in sign; the origin of this can be seen by commuting Eqs. (6a), (6b) with $\overline{Q}_\pm$, $Q_\pm$ respectively and using

$$[q_\pm, \overline{Q}_\pm] = [\overline{q}_\pm^0, Q_\pm] = 0 \tag{9}$$

(which follows from

$$[q_\pm, \partial_\pm] = [\overline{q}_\pm^0, \partial_\pm] = 0) \tag{10}$$

in conjunction with

$$[A, [B, C]] + [B, [A, C]] = [\{A, B\}, C] \tag{11}$$

and Eq. (4).

The transformations of Φ, $\overline{\Phi}$ induced by $\epsilon^+ Q_+ + \epsilon^- Q_- + \overline{\epsilon}^+ \overline{Q}_+ + \overline{\epsilon}^- \overline{Q}_-$ are then given by

$$\begin{aligned} \delta\Phi &= [\epsilon^+ Q_+ + \epsilon^- Q_- + \overline{\epsilon}^+ \overline{Q}_+ + \overline{\epsilon}^- \overline{Q}_-, \Phi], \quad (12)\\ \delta\overline{\Phi} &= [\epsilon^+ Q_+ + \epsilon^- Q_- + \overline{\epsilon}^+ \overline{Q}_+ + \overline{\epsilon}^- \overline{Q}_-, \overline{\Phi}], \quad (13) \end{aligned}$$

which, in view of Eq. (5), entails

$$\begin{aligned} \delta\varphi &= \epsilon^+\psi_+ + \epsilon^-\psi_-, \\ \delta\psi_+ &= \epsilon^- F + i\overline{\epsilon}^+\partial_+\varphi, \\ \delta\psi_- &= -\epsilon^+ F + i\overline{\epsilon}^-\partial_-\varphi, \\ \delta F &= -i\overline{\epsilon}^+\partial_+\psi_- + i\overline{\epsilon}^-\partial_-\psi_+ \\ \delta\overline{\varphi} &= -\overline{\epsilon}^+\overline{\psi}_+ - \overline{\epsilon}^-\overline{\psi}_-, \\ \delta\overline{\psi}_+ &= -i\epsilon^+\partial_+\overline{\varphi} - \overline{\epsilon}^-\overline{F}, \\ \delta\overline{\psi}_- &= -i\epsilon^-\partial_-\overline{\varphi} + \overline{\epsilon}^+\overline{F}, \\ \delta\overline{F} &= i\epsilon^+\partial_+\overline{\psi}_- - i\epsilon^-\partial_-\overline{\psi}_+. \end{aligned} \tag{14}$$

By virtue of Eqs. (5), (6a), (6b) we can also write

$$\delta\varphi = (\epsilon^+ q_+ + \epsilon^- q_- + \overline{\epsilon}^+ \overline{q}_+ + \overline{\epsilon}^- \overline{q}_-)\varphi, \tag{15}$$

with similar expressions for the other component fields.

The effect of the $\int d^2\theta d^2\overline{\theta}$ in Eq. (1) is to yield the component action as the $\theta^2\overline{\theta}^2$ term in the expansion of $K(\Phi, \overline{\Phi})$, giving

$$\begin{aligned} S_0 &= \int d^2x \Big[K_{\overline{j}}\partial_+\partial_-\overline{\varphi}^{\overline{j}} + K_{\overline{j}\overline{k}}\partial_+\overline{\varphi}^{\overline{j}}\partial_-\overline{\varphi}^{\overline{k}} \\ &+ K_{i\overline{j}}\Big(i\psi_+^i\partial_-\overline{\psi}_+^{\overline{j}} + i\psi_-^i\partial_+\overline{\psi}_-^{\overline{j}} + F^i\overline{F}^{\overline{j}} \Big) \\ &- K_{ik\overline{j}}\psi_+^i\psi_-^k\overline{F}^{\overline{j}} - K_{\overline{i}\overline{k}j}\overline{\psi}_+^{\overline{i}}\overline{\psi}_-^{\overline{k}}F^j + iK_{i\overline{j}\overline{k}}\Big(\psi_+^i\overline{\psi}_+^{\overline{j}}\partial_-\overline{\varphi}^{\overline{k}} + \psi_-^i\overline{\psi}_-^{\overline{j}}\partial_+\overline{\varphi}^{\overline{k}} \Big) \\ &+ K_{ij\overline{i}\overline{j}}\psi_+^i\psi_-^j\overline{\psi}_+^{\overline{i}}\overline{\psi}_-^{\overline{j}} \Big], \end{aligned} \tag{16}$$

where $K_i = \frac{\partial K}{\partial \varphi^i}$. It is easily verified using Eqs. (7), (16), that

$$q_{\pm} S_0 = \overline{q}^0_{\pm} S_0 = 0, \tag{17}$$

which demonstrates the invariance of the action under supersymmetry transformations (according to Eq. (15)).

The action Eq. (16) can also be written using the operators $q_{\pm}$, $\overline{q}^0_{\pm}$ as

$$S_0 = \int d^2x q_- q_+ \overline{q}^0_- \overline{q}^0_+ K; \tag{18}$$

which of course guarantees Eq. (17) due to the nilpotency of $q_{\pm}$, $\overline{q}^0_{\pm}$, which in turn follows from that of $Q_{\pm}$, $\overline{Q}_{\pm}$ in Eq. (4). There is something intriguingly reminiscent of the BRST formalism in the use of nilpotent operators to obtain an invariant expression. It is worth mentioning that after eliminating the auxiliary fields F, $\overline{F}$ using their equations of motion, the action may be written in the form

$$\begin{aligned} S_0 &= \int d^2x \Big[g_{i\bar{j}} \left(\partial_+ \varphi^i \partial_- \overline{\varphi}^{\bar{j}} + i \psi^i_+ \partial_- \overline{\psi}^{\bar{j}}_+ + i \psi^i_- \partial_+ \overline{\psi}^{\bar{j}}_- \right) \\ &+ R_{i\bar{i}j\bar{j}} i \psi^i_+ \psi^j_- \overline{\psi}^{\bar{i}}_+ \overline{\psi}^{\bar{j}}_- \Big] \end{aligned} \tag{19}$$

where $R_{i\bar{i}j\bar{j}}$ is the Riemann tensor constructed from the Kähler metric $g_{i\bar{j}} \equiv K_{i\bar{j}}$. This form is manifestly generally covariant with respect to this metric.

At the quantum level the renormalisation of the model may be achieved by replacing the classical Kähler potential by a bare version, K_B, chosen so as to cancel the ultraviolet divergences order by order. Using the standard dimensional regularisation with the spacetime dimension continued to $d = 2 - \epsilon$, at one loop we have simply

$$K_B = K + \frac{1}{2\pi\epsilon} \mathrm{tr} \ln K_{i\bar{j}}. \tag{20}$$

This corresponds to replacing the Kähler metric by

$$g_{Bi\bar{j}} = g_{i\bar{j}} + \frac{1}{2\pi\epsilon} R_{i\bar{j}} \tag{21}$$

where $R_{i\bar{j}}$ is the Ricci tensor. As mentioned in the introduction, the next divergence appears at the four-loop level [29, 30]. Just as the classical action may be obtained by the operators $q_{\pm}$, $\overline{q}^0_{\pm}$ acting on K as in Eq. (18), we may write

$$S_{0B} = \int d^2x q_- q_+ \overline{q}^0_- \overline{q}^0_+ K_B, \tag{22}$$

so that in particular $q_- q_+ \overline{q}^0_- \overline{q}^0_+ \mathrm{tr} \ln K_{i\bar{j}}$ has the effect of reproducing the one-loop divergences, in a somewhat compact form.

3 Non-anticommutative supersymmetry in two dimensions

In this section we repeat the analysis of the previous section for the case of deformed two-dimensional supersymmetry. For the deformed version we take

$$(\theta^{\pm})^2 = (\overline{\theta}^{\pm})^2 = 0, \quad \{\overline{\theta}^{+}, \overline{\theta}^{-}\} = 0, \quad \{\theta^{+}, \theta^{-}\} = \frac{1}{M}. \tag{23}$$

The charges then satisfy the algebra

$$\begin{aligned} Q_+^2 = Q_-^2 &= \{Q_+, Q_-\} = 0, \\ \overline{Q}_+^2 = \overline{Q}_-^2 = 0, \qquad & \{\overline{Q}_+, \overline{Q}_-\} = -\frac{4}{M}\frac{\partial^2}{\partial y^+ \partial y^-}, \\ \{\overline{Q}_+, Q_+\} = -i\partial_+, \qquad & \{\overline{Q}_-, Q_-\} = -i\partial_-. \end{aligned} \tag{24}$$

The non-anticommutativity is implemented at the level of superfields by introducing the star-product, which satisfies

$$\begin{aligned} \theta^+ * \theta^- = \theta^+\theta^- + \frac{1}{2M}, \qquad & \theta^- * \theta^+ = -\theta^+\theta^- + \frac{1}{2M}, \\ \theta^+ * \theta^+\theta^- = -\frac{1}{2M}\theta^+, \qquad & \theta^- * \theta^+\theta^- = \frac{1}{2M}\theta^-, \\ \theta^+\theta^- * \theta^+\theta^- &= \frac{1}{4M^2}. \end{aligned} \tag{25}$$

We now wish to construct differential operators $\overline{q}_{\pm}$ representing the effects of $\overline{Q}_{\pm}$ in the deformed case in a similar manner to Eq. (6a,6b), extending $\overline{q}^0_{\pm}$ given in Eq. (7) for the undeformed case. (The operators $q_{\pm}$ are unchanged by the deformation.) We start by examining the effects of $\overline{Q}_{\pm}$ on powers of Φ alone, since mixed products of Φ and $\overline{\Phi}$ present additional complications. Defining

$$I_r^{(n)} = \int_{-\frac{1}{2}}^{\frac{1}{2}} d\xi \left(\frac{\xi}{M}\right)^r \left(\varphi + \frac{\xi}{M}F\right)^n \tag{26}$$

it is straightforward to show using the methods of Ref. [27] that

$$\Phi_*^n = (1 + \theta^+ q_+)(1 + \theta^- q_-)\left(I_0^{(n)} - q_+ q_- I_1^{(n)}\right), \tag{27}$$

where Φ_*^n denotes the star-product of n Φ's. Then acting on Φ_*^n, $\overline{Q}_{\pm}$ are represented by

$$\begin{aligned} \overline{q}_+^{\Phi} &= \overline{q}_+^0 - \frac{i}{2M}\partial_+ q_- + i(-q'_+ q'_-[\partial'_+ q'_-]\tilde{\mathcal{O}} + \partial'_+ q'_- \mathcal{O} + [\partial'_+ q'_-]\mathcal{O}), \\ \overline{q}_-^{\Phi} &= \overline{q}_-^0 - \frac{i}{2M}\partial_- q_+ - i(-q'_+ q'_-[\partial'_- q'_+]\tilde{\mathcal{O}} + \partial'_- q'_+ \mathcal{O} + [\partial'_- q'_+]\mathcal{O}). \end{aligned} \tag{28}$$

Here a prime denotes the part of the operator containing derivatives with respect to the chiral (but not the anti-chiral) fields, and correspondingly

$$\partial'_\pm = \partial_\pm\varphi\frac{\partial}{\partial\varphi} + \partial_\pm\psi_+\frac{\partial}{\partial\psi_+} + \partial_\pm\psi_-\frac{\partial}{\partial\psi_-} + \partial_\pm F\frac{\partial}{\partial F}. \tag{29}$$

Moreover,

$$[\partial'_+ q'_-] = \partial_+\psi_-\frac{\partial}{\partial\varphi} + \partial_+ F\frac{\partial}{\partial\psi_+}, \tag{30}$$

and

$$\begin{aligned}\mathcal{O}I_0^{(n)} &= I_1^{(n)},\\ \mathcal{O}I_1^{(n)} &= I_2^{(n)} - \tilde{\mathcal{O}}I_0^{(n)}.\end{aligned} \tag{31}$$

These properties are guaranteed by the following definitions:

$$\begin{aligned}\mathcal{O} &= \sum_{r=1}^{\infty} a_r\left(\frac{1}{M^2}\right)^r\left(F\frac{\partial}{\partial\varphi}\right)^{2r-1},\\ \tilde{\mathcal{O}} &= \sum_{r=1}^{\infty}(2r-1)a_r\left(\frac{1}{M^2}\right)^r\left(F\frac{\partial}{\partial\varphi}\right)^{2r-2},\end{aligned} \tag{32}$$

where the a_r must satisfy for each $n \geq 1$

$$\sum_{r=0}^{n-1}\frac{a_{n-r}}{2^{2r}(2r+1)(2r)!} = \frac{1}{2^{2n}(2n+1)(2n-1)!}. \tag{33}$$

We have been unable to find a closed form for the a_r; the first few, determined recursively, being

$$a_1 = \frac{1}{12}, \quad a_2 = -\frac{1}{720}, \quad a_3 = \frac{1}{2^5.3^3.5.7}. \tag{34}$$

To check that the operators in Eq. (28) do indeed represent the operators $\overline{Q}_\pm$ according to

$$[\overline{Q}_\pm, \Phi_*^n]_* = \overline{q}_\pm^\Phi \Phi_*^n \tag{35}$$

(where $[\ ,\ ]_*$ represents the commutator evaluated using star-products) we need to use Eqs. (31) in conjunction with

$$\begin{aligned}\overline{q}_+^{0\prime}I_r^{(n)} = -i[\partial'_+ q'_-]I_{r+1}^{(n)}, &\qquad \overline{q}_-^{0\prime}I_r^{(n)} = i[\partial'_- q'_+]I_{r+1}^{(n)},\\ q''_+ I_0^{(n)} = q''_- I_0^{(n)} &= q''_+ I_1^{(n)} = q''_- I_1^{(n)} = 0.\end{aligned} \tag{36}$$

where a double prime denotes the part of the operator containing derivatives with respect to the anti-chiral (but not the chiral) fields.

It is easy to check that the operators in Eq. (28) satisfy the anticommutation relations of Eq. (24), using

$$\left[\overline{q}_\pm^0, F\frac{\partial}{\partial\varphi}\right] = \mp i[\partial'_\pm q'_\mp] \tag{37}$$

(which implies

$$[\overline{q}^0_\pm, \mathcal{O}] = \mp i[\partial'_\pm q'_\mp]\tilde{\mathcal{O}}). \tag{38}$$

When acting on products of both Φ and $\overline{\Phi}$ the situation is more complicated, and the operators representing $\overline{Q}_\pm$ will require modification. We have $\overline{\Phi}^n_* = \overline{\Phi}^n$ and we find

$$\begin{aligned} \Phi^n_* * \overline{\Phi}^m &= (1+\theta^+ q_+)(1+\theta^- q_-) \\ &\quad \left[1 - \overline{\theta}^+ \left(\overline{q}^{0\prime\prime}_+ - \frac{i}{2M}\partial''_+ q'_-\right)\right] \left[1 - \overline{\theta}^- \left(\overline{q}^{0\prime\prime}_- - \frac{i}{2M}\partial''_- q'_+\right)\right] \\ &\quad \left(I^{(n)}_0 - q_+ q_- I^{(n)}_1\right) \overline{\varphi}^m. \end{aligned} \tag{39}$$

We then have

$$\begin{aligned} \left[\overline{Q}_+, \Phi^n_* * \overline{\Phi}^m\right]_* &= \left\{\overline{q}^\Phi_+ - \frac{i}{2M}(\partial''_+ q'_- - \partial'_+ q''_-)\right\} \Phi^n_* * \overline{\Phi}^m, \\ \left[\overline{Q}_-, \Phi^n_* * \overline{\Phi}^m\right]_* &= \left\{\overline{q}^\Phi_- - \frac{i}{2M}(\partial''_- q'_+ - \partial'_- q''_+)\right\} \Phi^n_* * \overline{\Phi}^m. \end{aligned} \tag{40}$$

On the other hand we have

$$\begin{aligned} \overline{\Phi}^m * \Phi^n_* &= (1+\theta^+ q_+)(1+\theta^- q_-) \\ &\quad \left[1 - \overline{\theta}^+ \left(\overline{q}''_+ + \frac{i}{2M}\partial''_+ q'_-\right)\right] \left[1 - \overline{\theta}^- \left(\overline{q}''_- + \frac{i}{2M}\partial''_- q'_+\right)\right] \\ &\quad \left(I^{(n)}_0 - q_+ q_- I^{(n)}_1\right) \overline{\varphi}^m, \end{aligned} \tag{41}$$

and correspondingly

$$\begin{aligned} \left[\overline{Q}_+, \overline{\Phi}^m * \Phi^n_*\right]_* &= \left\{\overline{q}^\Phi_+ + \frac{i}{2M}(\partial''_+ q'_- - \partial'_+ q''_-)\right\} \overline{\Phi}^m * \Phi^n_*, \\ \left[\overline{Q}_-, \overline{\Phi}^m * \Phi^n_*\right]_* &= \left\{\overline{q}^\Phi_- + \frac{i}{2M}(\partial''_- q'_+ - \partial'_- q''_+)\right\} \overline{\Phi}^m * \Phi^n_*. \end{aligned} \tag{42}$$

We see from Eqs. (40), (42) that the operators representing $\overline{Q}_\pm$ are modified in different ways depending on whether they act on $\Phi^n_* * \overline{\Phi}^m$ or $\overline{\Phi}^m * \Phi^n_*$. It is unusual to find that the representation of the operator depends on the ordering of the term on which it acts. However, fortunately we are only interested in the deformed version of the Kähler potential, in which each term should be defined as a symmetrised star-product of Φ's and $\overline{\Phi}$'s, and therefore the ordering question will not arise. For such a symmetrised product, the representations of $\overline{Q}_\pm$ will again be different from those given in Eq. (28). For an undeformed Kähler potential

$$K[\Phi, \overline{\Phi}] = \sum_{n,m} K_{n,m} \Phi^n \overline{\Phi}^m, \tag{43}$$

the natural definition of the deformed Kähler potential is

$$K_*[\Phi, \overline{\Phi}] = \sum_{n,m} K_{n,m} [\Phi^n \overline{\Phi}^m]_*, \tag{44}$$

where $[\Phi^n\overline{\Phi}^m]_*$ represents the symmetrised star-product of n Φ's and m $\overline{\Phi}$'s. It can be shown that

$$\begin{aligned} K_*[\Phi,\overline{\Phi}] &= (1+\theta^+q_+)(1+\theta^-q_-)\left(1-\overline{\theta}^+\overline{q}_+^{0\prime\prime}\right)\left(1-\overline{\theta}^-\overline{q}_-^{0\prime\prime}\right) \\ & \quad [K_0(\varphi,F,\overline{\varphi})-q_+q_-K_1(\varphi,F,\overline{\varphi})] \\ &- \frac{1}{4M^2}\overline{\theta}^+\overline{\theta}^-q'_+q'_-\partial''_+\partial''_-K_0(\varphi,F,\overline{\varphi}), \end{aligned} \tag{45}$$

where

$$K_m(\varphi,F,\overline{\varphi}) = \int_{-\frac{1}{2}}^{\frac{1}{2}} d\xi\xi^m K\left(\varphi+\frac{\xi}{M}F,\overline{\varphi}\right). \tag{46}$$

The symmetrisation has resulted in the disappearance of most of the terms involving a $\frac{1}{2M}$ in Eqs. (39), (41). Correspondingly we no longer need the $\frac{1}{2M}$ terms in Eqs. (40), (42). However, the residual $\frac{1}{4M^2}$ term requires a modification of the operators given in Eqs. (28), so that

$$\begin{aligned} \overline{q}_+ &= \overline{q}_+^0 - \frac{i}{2M}\partial_+q_- - \frac{i}{4M^2}(\partial''_+q'_+q'_-q''_- + \partial'_+q'_-q''_+q''_-) \\ &+ i(-q'_+q'_-[\partial'_+q'_-]\tilde{\mathcal{O}} + \partial'_+q'_-\mathcal{O} + [\partial'_+q'_-]\mathcal{O}), \\ \overline{q}_- &= \overline{q}_-^0 - \frac{i}{2M}\partial_-q_+ + \frac{i}{4M^2}(-\partial''_-q'_+q'_-q''_+ + \partial'_-q'_+q''_+q''_-) \\ &- i(-q'_+q'_-[\partial'_-q'_+]\tilde{\mathcal{O}} + \partial'_-q'_+\mathcal{O} + [\partial'_-q'_+]\mathcal{O}), \end{aligned} \tag{47}$$

We can verify that these operators do indeed implement the operators $\overline{Q}_\pm$ according to

$$[\overline{Q}_\pm, K_*]_* = \overline{q}_\pm K_*, \tag{48}$$

using the analogue of Eq. (31) for the Kähler potential,

$$\begin{aligned} \mathcal{O}K_0 &= K_1, \\ \mathcal{O}K_1 &= K_2 - \tilde{\mathcal{O}}K_0 \end{aligned} \tag{49}$$

together with the analogue of Eq. (36),

$$\begin{aligned} \overline{q}'_+K_r = -i[\partial'_+q'_-]K_{r+1}, &\qquad \overline{q}'_-K_r = i[\partial'_-q'_+]K_{r+1}, \\ q''_+K_0 = q''_-K_0 &= q''_+K_1 = q''_-K_1 = 0. \end{aligned} \tag{50}$$

The action is given by the $\theta^2\overline{\theta}^2$ term and hence from Eq. (45)

$$S = \int d^2x q_-q_+\overline{q}''_-\overline{q}''_+(K_0 - q_+q_-K_1), \tag{51}$$

which can be expanded as [25] - [27]

$$\begin{aligned} S &= \int d^2x\Big\{\partial_{\overline{j}}K_0\partial_+\partial_-\overline{\varphi}^{\overline{j}} + \partial_{\overline{j}}\partial_{\overline{k}}K_0\partial_+\overline{\varphi}^{\overline{j}}\partial_-\overline{\varphi}^{\overline{k}} + \partial_i\partial_{\overline{j}}K_0\Big(i\psi^i_+\partial_-\overline{\psi}^{\overline{j}}_+ + i\psi^i_-\partial_+\overline{\psi}^{\overline{j}}_- + F^i\overline{F}^{\overline{j}}\Big) \\ &- \partial_i\partial_k\partial_{\overline{j}}K_0\psi^i_+\psi^k_-\overline{F}^{\overline{j}} - \partial_{\overline{i}}\partial_{\overline{k}}\partial_jK_0\overline{\psi}^{\overline{i}}_+\overline{\psi}^{\overline{k}}_-F^j + i\partial_i\partial_{\overline{j}}\partial_{\overline{k}}K_0\Big(\psi^i_+\overline{\psi}^{\overline{j}}_+\partial_-\overline{\varphi}^{\overline{k}} + \psi^i_-\overline{\psi}^{\overline{j}}_-\partial_+\overline{\varphi}^{\overline{k}}\Big) \\ &+ \partial_i\partial_j\partial_{\overline{i}}\partial_{\overline{j}}K_0\psi^i_+\psi^j_-\overline{\psi}^{\overline{i}}_+\overline{\psi}^{\overline{j}}_- + \frac{1}{M}\Big(\partial_i\partial_{\overline{j}}K_1F^i\partial_+\partial_-\overline{\varphi}^{\overline{j}} - \partial_i\partial_k\partial_{\overline{j}}K_1\psi^i_+\psi^k_-\partial_+\partial_-\overline{\varphi}^{\overline{j}} \\ &+ \partial_i\partial_{\overline{j}}\partial_{\overline{k}}K_1F^i\partial_+\overline{\varphi}^{\overline{j}}\partial_-\overline{\varphi}^{\overline{k}} - \partial_i\partial_k\partial_{\overline{j}}\partial_{\overline{k}}K_1\psi^i_+\psi^k_-\partial_+\overline{\varphi}^{\overline{j}}\partial_-\overline{\varphi}^{\overline{k}}\Big)\Big\}. \end{aligned} \tag{52}$$

It can then be checked that also

$$S = \int d^2x q_- q_+ \overline{q}_- \overline{q}_+ (K_0 - q_+ q_- K_1) = \int d^2x q_- q_+ \overline{q}_- \overline{q}_+ K_0. \tag{53}$$

Note that in Eq. (51), the K_1 term is indispensable and is entirely responsible for the K_1 terms in Eq. (52); while in Eq. (53), the K_1 term is redundant and can be omitted, leading to a form for the action similar to Eq. (18) in the undeformed case. The K_1 terms in Eq. (52) are generated from Eq. (53) by applying Eq. (49).

Finally, from Eq. (53), we see that (as in the undeformed case) the nilpotency of $q_\pm$, $\overline{q}_\pm$, which follows from that of $Q_\pm$, $\overline{Q}_\pm$ in Eq. (24), ensure

$$q_\pm S = \overline{q}_\pm S = 0; \tag{54}$$

so that the deformed action is invariant under the action of $q_\pm$ and $\overline{q}_\pm$.

4 One-loop corrections

Our goal was to investigate the one-loop corrections for the deformed theory, and see whether they could be interpreted in terms of a "smearing" of the background geometry as at the classical level. It seemed reasonable to do this order by order in $\frac{1}{M^2}$. (Note that K_i is a power series in $\frac{1}{M^2}$, starting at $\frac{1}{M^0}$ for i even and $\frac{1}{M}$ for i odd). We then had to make a choice of method, since the computation of the one-loop and higher quantum corrections for the undeformed Kähler σ-model may be performed in several different ways. The superspace computation [29] is the most efficient, though it has the disadvantage that it conceals the generally covariant form of the results, i.e. that they can be expressed in terms of the Kähler metric and its associated Riemann tensor in a generally covariant way. The covariant form of the classical action is achieved in the component formulation upon integrating out the auxiliary fields, and computations up to four loops have also been carried out in this formalism [30]. Superspace computations in the non-anticommutative case have been performed in the four-dimensional context [20], [21] but the formalism is technically rather complex; on the other hand, integrating out the auxiliary fields in the deformed action Eq. (52) would be difficult and in any case it is no longer clear if general covariance is a useful guide.

Accordingly, we decided to perform the calculation in the uneliminated component formulation. However, it rapidly becomes apparent that there is a plethora of diagrams to consider. We started by computing the divergences for the set of graphs with a single insertion of a vertex with a $\frac{1}{M^2}$ factor derived from a K_1 term in Eq. (52). We then realised that the divergent contributions from this set of graphs (numbering about 200) could be expressed much more concisely as $q_- q_+ \mathcal{K}$ for some $\mathcal{K}$ (which we call a kernel). With this as a guide, we were then able to construct the corresponding $\mathcal{K}$ for the full set of one-loop $\frac{1}{M^2}$ diagrams, explicitly computing only a small subset of these to serve as a check. Of course, this is reminiscent of the fact remarked on earlier that in the undeformed case the one-loop quantum corrections may be written in terms of $q_- q_+ \overline{q}^0_- \overline{q}^0_+ \operatorname{tr} \ln K_{i\bar{j}}$. The full kernel, $\mathcal{K}^{(1)}_B$, is displayed in the Appendix using a convenient

diagrammatic notation. It is tempting to wonder if the analogy with the undeformed case goes further so that we may write

$$S_B^{(1)} = \int d^2x q_- q_+ \overline{q}_- \overline{q}_+ \tilde{\mathcal{K}}_B^{(1)} \tag{55}$$

for some underlying $\tilde{\mathcal{K}}_B^{(1)}$, where $\overline{q}_\pm$ are the deformed operators constructed in Eq. (47); indeed this was our motivation for constructing these operators in the first instance. Unfortunately this turns out not to be the case, as is easily seen: focussing on the set of graphs in $\mathcal{K}_B^{(1)}$ with five vertices, four with a single fermion and one with an F, it can be seen that the graphs with six vertices, five with one fermion and one with an F, (and no derivatives) created by the action of $\overline{q}_+$ on this set do not cancel. In drawing this conclusion we can restrict attention to the effect of $\overline{q}_+^0$ since the remaining terms in $\overline{q}_+$ all contain derivatives. Since this is the only source of graphs of this type in $\overline{q}_+\mathcal{K}_B^{(1)}$, we see that $\overline{q}_+\mathcal{K}_B^{(1)} \neq 0$ (and by the same token $\overline{q}_-\mathcal{K}_B^{(1)} \neq 0$). Therefore $q_-q_+\overline{q}_+\mathcal{K}_B^{(1)} \neq 0$ and $q_-q_+\overline{q}_-\mathcal{K}_B^{(1)} \neq 0$) (consider for instance those graphs for which q_-q_+ simply attaches an F at the vertex already containing an F); and so $\overline{q}_+S_B^{(1)} \neq 0$, $\overline{q}_-S_B^{(1)} \neq 0$. This immediately implies (due once again to the nilpotency of $\overline{q}_\pm$) that $S_B^{(1)}$ cannot be of the form Eq. (55). It is noteworthy that the classical behaviour is not reproduced at the quantum level, and in particular that the one-loop effective action is not invariant under $\overline{q}_\pm$, even though the classical action was.

5 Conclusions

We have constructed differential operators which express the non-anticommutative supersymmetry according to

$$[Q_\pm, \Phi] = q_\pm \Phi, \quad [\overline{Q}_\pm, \Phi] = \overline{q}_\pm \Phi \tag{56}$$

and which therefore reproduce the deformed algebra in Eq. (24). It then follows from the fact that the classical action may be written $S = \int d^2x q_- q_+ \overline{q}_- \overline{q}_+ K_0$ and the nilpotency of $q_\pm$, $\overline{q}_\pm$ that $q_\pm S = \overline{q}_\pm S = 0$. However, we then examined the one-loop effective action and showed that although we could express the one-loop divergences as

$$S_B^{(1)} = \int d^2x q_- q_+ \mathcal{K}_B^{(1)}, \tag{57}$$

it was not possible in turn to write $\mathcal{K}_B^{(1)} = \overline{q}_-\overline{q}_+\tilde{\mathcal{K}}_B^{(1)}$ for some $\tilde{\mathcal{K}}_B^{(1)}$. Correspondingly, although $q_\pm S_B^{(1)} = 0$, $\overline{q}_\pm S_B^{(1)} \neq 0$. In fact, an invariance of the classical action can be shown to lead directly to an invariance of the quantum effective action only in simple cases, namely for linear transformations of the fields; such as, indeed, the transformations corresponding to $q_\pm$. In the case of non-linear transformations, the transformation properties of the effective action are expressed through Ward identities. In the case at

hand, the effect of $\overline{q}_\pm$ on a single field is in fact linear, though the effect on functions of the fields is more complicated.

The fact that $q_\pm S = 0$ implies $q_\pm S_B^{(1)} = 0$ is therefore easy to understand. However, it would also be interesting to try to prove to all orders the stronger statement, that $S_B = \int d^2x q_- q_+ \mathcal{K}_B$ for an appropriate K_B, which we have shown to be valid at one loop and first order in $\frac{1}{M^2}$. Our original motivation in embarking on this calculation was to see if the "smearing" of the classical geometry was mirrored at the quantum level. This seems unlikely in view of the non-renormalisability of the theory, manifested here by the appearance of divergent terms in the one-loop effective action with, for instance, 6 fermion fields; and in fact one can obtain divergent diagrams with arbitrary numbers of external legs by inserting chains of deformed vertices of indefinite length into appropriate "propagators" in a given divergent diagram. The $\mathcal{N} = \frac{1}{2}$ gauge theory in four dimensions, albeit power-counting non-renormalisable, turned out to have only a finite number of types of counterterm. This property is associated with the non-hermiticity of the theory, a generic feature of these deformed supersymmetric theories; but in the four-dimensional case this can be codified as a kind of R-parity [16] which severely restricts the types of counterterm; presumably such an effect is absent in two dimensions. The combination of non-renormalisability and the novel form of the invariance seems likely to preclude the possibility of obtaining a succinct form of the quantum effective action which could be interpreted in terms of a modification of the (smeared) background geometry, though it would be interesting to investigate this further. Of course, although we committed ourselves to working in the component formulation, believing the superspace computation of quantum corrections to be very unwieldy in the nonanticommutative case, this alternative might be worth pursuing to see if a simpler form of the results might be achieved thereby.

Acknowledgements

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A One-Loop Kernel

We present here in largely diagrammatic form the kernel $\mathcal{K}_B^{(1)}$ for the one-loop divergences, which are then given by $q_- q_+ \mathcal{K}_B^{(1)}$. Since $\overline{q}_- \overline{q}_+ K_0 = \overline{F}^{\overline{i}} K_{0\overline{i}} - \overline{\psi}_+^{\overline{i}} \overline{\psi}_-^{\overline{j}} K_{0\overline{i}\overline{j}}$, with a similar expression for K_1, the action in Eq. (51), and hence the kernel, separates into four sections which can separately be written as $q_- q_+$ acting on a kernel. The kernel may accordingly be written

$$\mathcal{K}_B^{(1)} = \frac{\partial^2 L_{M^2}}{\partial F^i \partial \overline{\varphi}^{\overline{j}}} K^{i\overline{j}} - \frac{\partial^2 L_{M^2}}{\partial F^i \partial F^j} K^{i\overline{k}} K^{j\overline{l}} (K_{\overline{k}\overline{l}m} F^m - K_{\overline{k}\overline{l}mn} \psi^m \psi^n) + \frac{1}{24 M^2} (A_1 + A_2 + 2A_3 + 2A_4), \tag{58}$$

where L_{M^2} is the M^2 term in the lagrangian of Eq. (52) and A_{1-4} are expressed diagrammatically below, in Figs. 3-8.

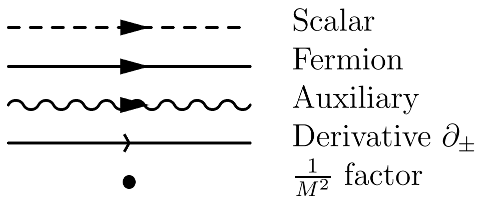

Figure 1: Figure conventions

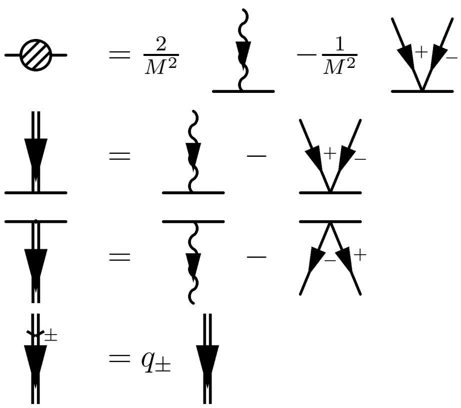

Figure 2: Shorthand notation for diagrams

In these diagrams a "propagator" in a loop denotes K^{-1} and vertices denote derivatives of K, while external lines attached to vertices represent the various fields according to the conventions in Fig. 1 and the convenient shorthand notations in Fig. 2.

Incoming (outgoing) arrows represent chiral (antichiral) fields, respectively. The ordering of fermion fields is fixed by the convention that we start at the left-most field at the top of the diagram and read clockwise around the loop. As an illustration of our notation, the first diagram in A_1 in Fig. 3 below represents

$$F^iF^jK_{ijk\overline{l}}K^{m\overline{l}}K^{k\overline{n}}(K_{m\overline{n}\overline{p}}\overline{F}^{\overline{p}} - K_{m\overline{n}\overline{p}\overline{q}}\overline{\psi}_+^{\overline{p}}\overline{\psi}_-^{\overline{q}}) \tag{59}$$

and the second represents

$$F^i\psi_+^j\psi_-^kK_{ip\overline{j}}K^{l\overline{j}}K_{jl\overline{k}}K^{m\overline{k}}(K_{m\overline{m}\overline{p}}\overline{F}^{\overline{p}} - K_{m\overline{m}\overline{p}\overline{q}}\overline{\psi}_+^{\overline{p}}\overline{\psi}_-^{\overline{q}})K^{n\overline{m}}K_{kn\overline{n}}K^{p\overline{n}} \tag{60}$$

(where $K^{i\overline{j}} \equiv K^{-1}_{i\overline{j}}$). Using $\partial_iK^{-1} = -K^{-1}\partial_iKK^{-1}$ the effect of $q_\pm$ is to add external lines and create new vertices. After acting on a diagram with q_-q_+, we obtain a set of diagrams which (unless they cancel with similar contributions from other kernel diagrams) correspond to viable one-loop Feynman graphs, the vertex with the dot or the "blob" being the one from the deformed part of the action, and hence with an accompanying $\frac{1}{M^2}$ factor.

We observe some intriguing patterns in the groups of diagrams appearing in A_{1-4} above. For instance, one group of terms in A_1 is repeated in A_4 with the simple substitution of a "blob" for an incoming F (and a factor of $\frac{1}{2}$); and another group of terms

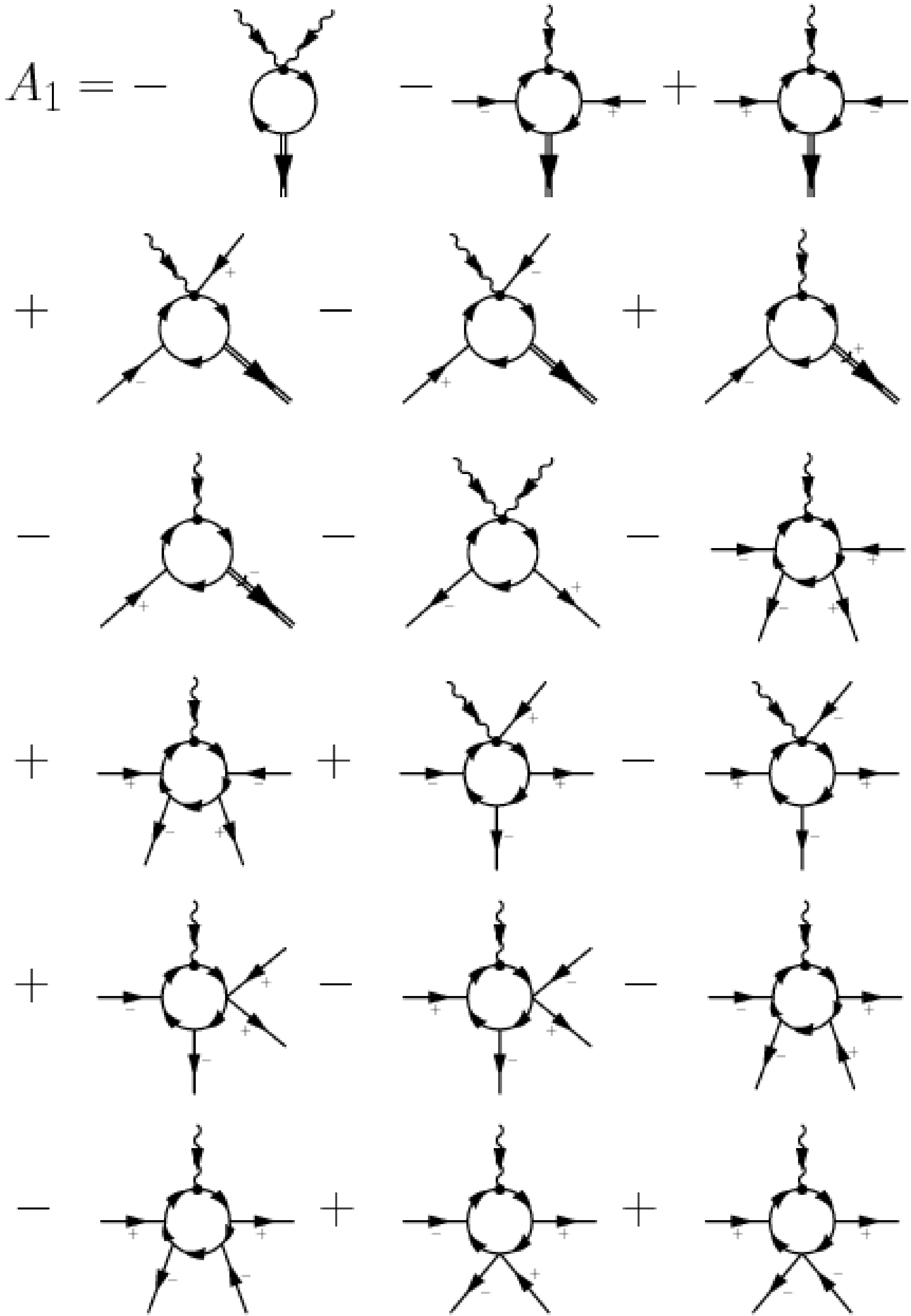

Figure 3: Diagrams for A_1

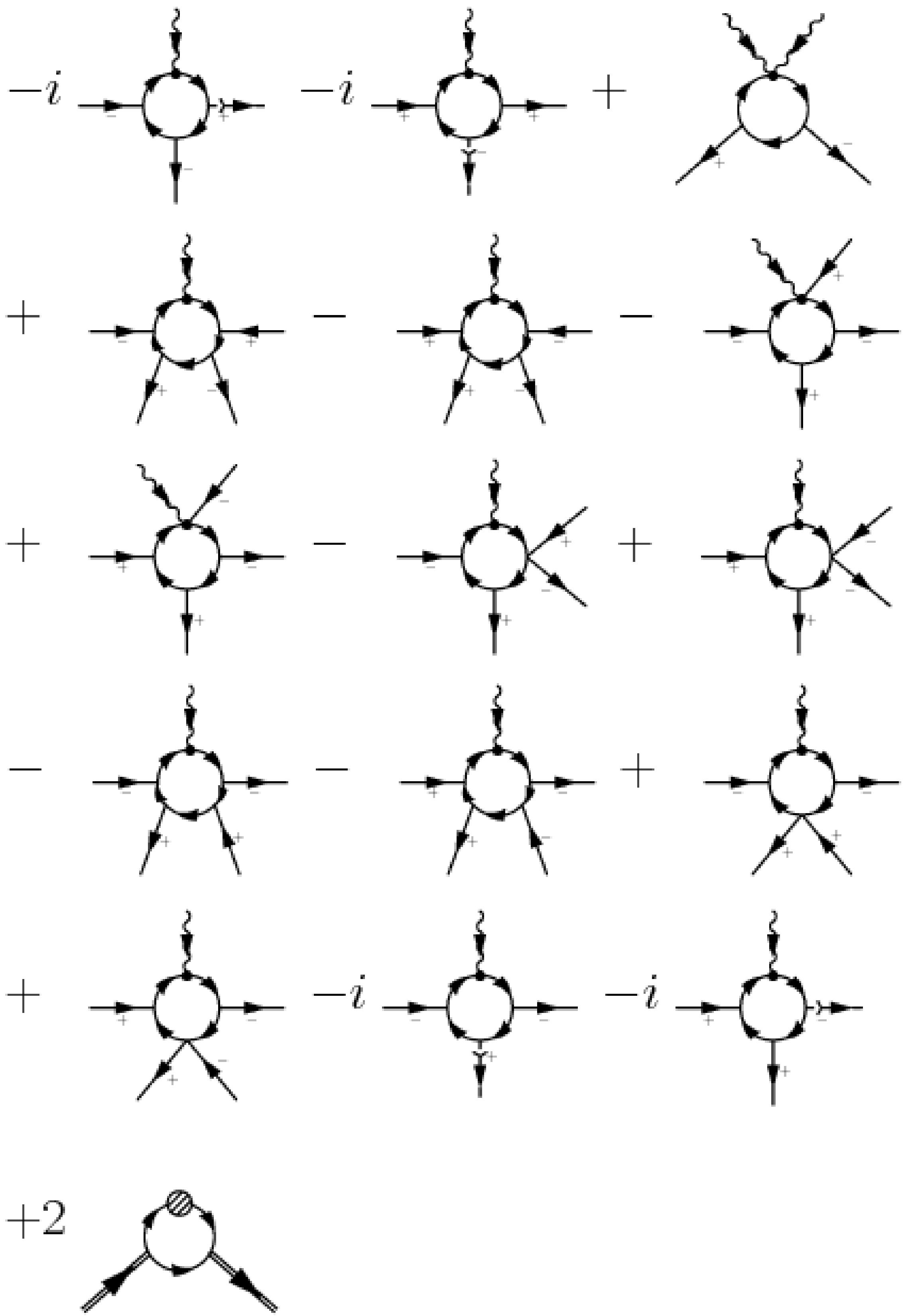

Figure 4: Diagrams for A_1 (continued)

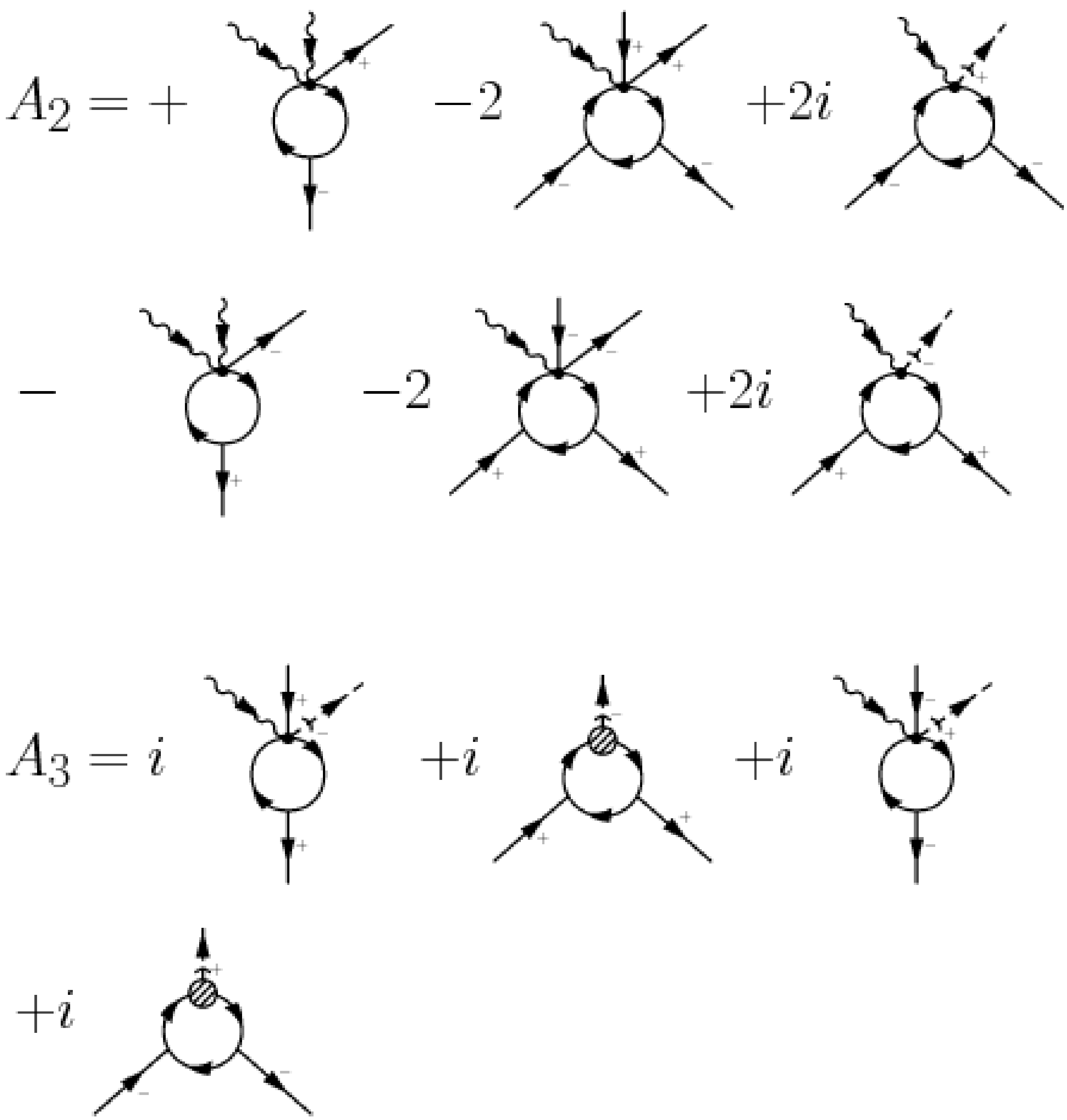

Figure 5: Diagrams for A_2 and A_3

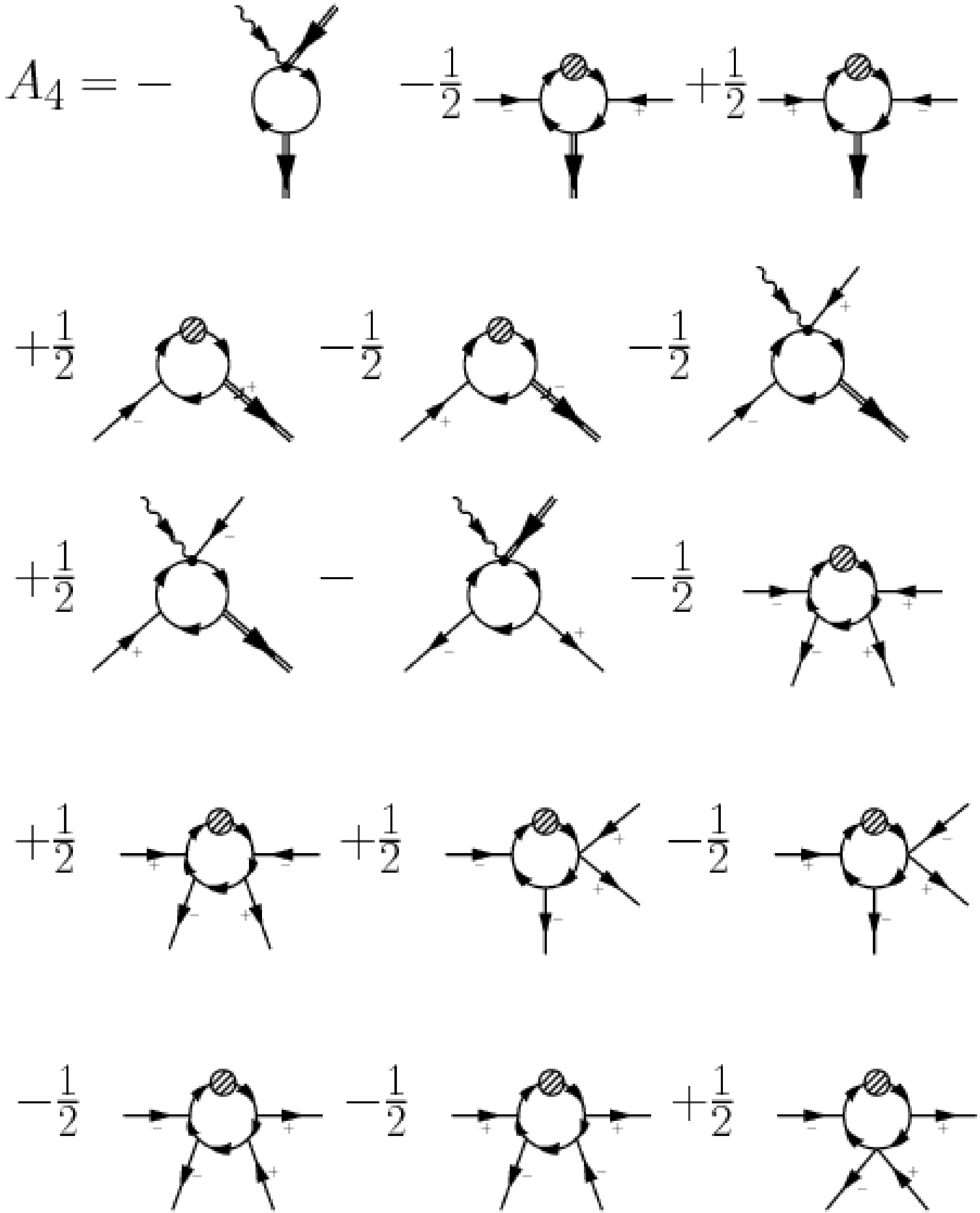

Figure 6: Diagrams for A_4

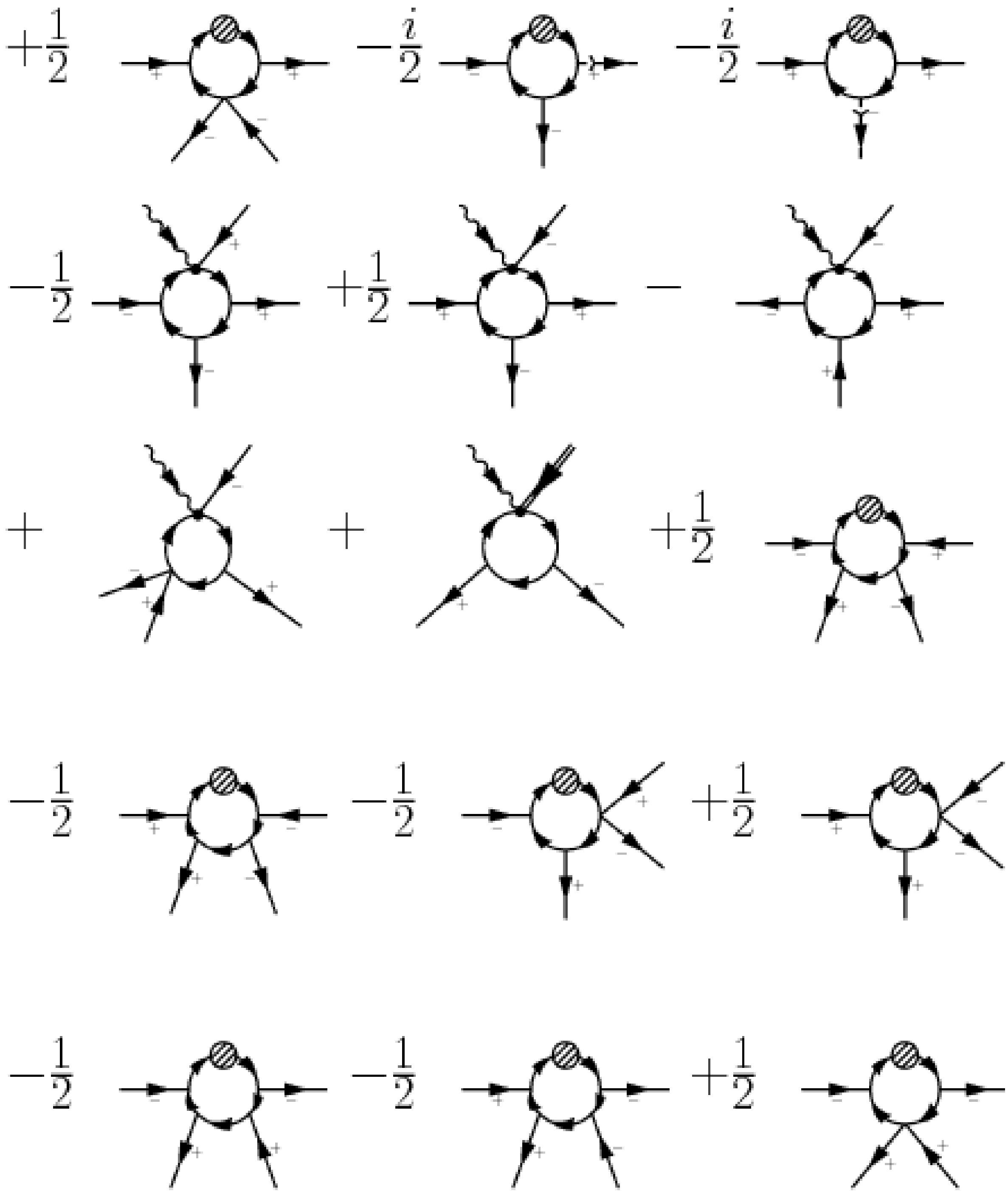

Figure 7: Diagrams for A_4 (continued)

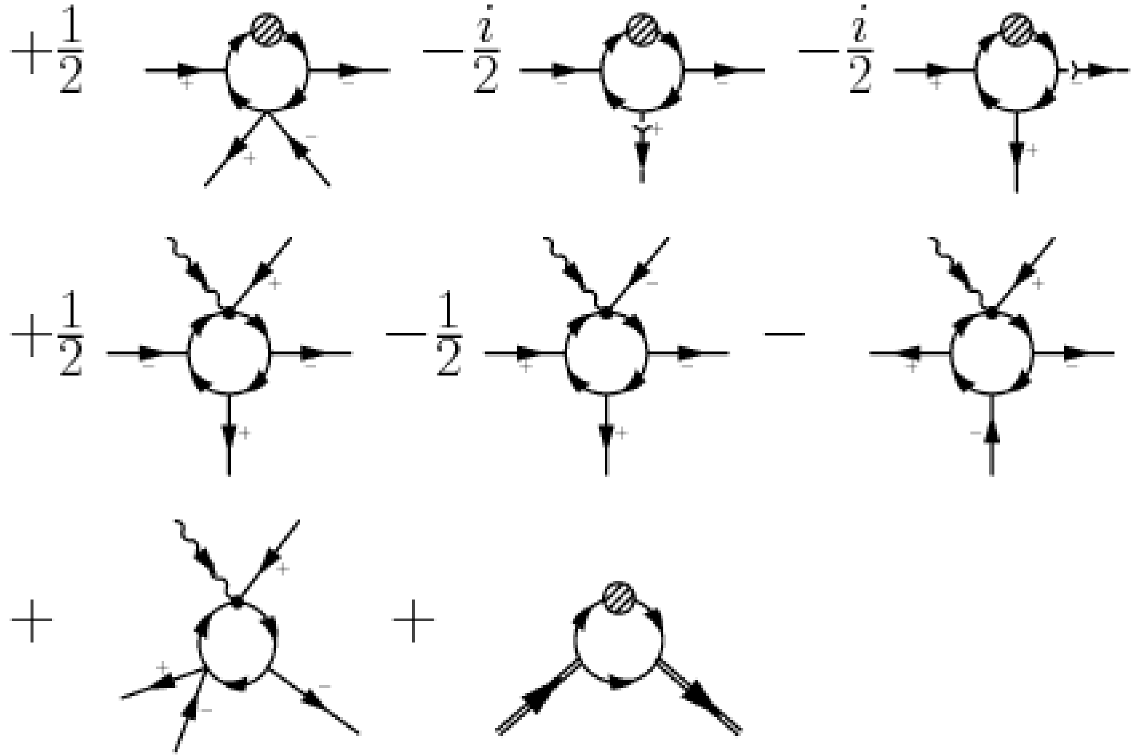

Figure 8: Diagrams for A_4 (continued)

in A_1 may be obtained from the former group in A_1 by replacing a $\overline{\psi}_+$ followed by an adjacent $\overline{\psi}_-$ (or a $\overline{\psi}_-$ followed by an adjacent $\overline{\psi}_+$) with a $\overline{F} - \overline{\psi}_+\overline{\psi}_-$ (i.e. an outgoing double line). Finally, the graphs in A_3 are similar to those of A_2.

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